PHYS 331 – Assignment #2

Due Monday, November 6 at 08:00

This assignment will be used to generalize the transmission line analysis that we did during the PHYS 331 lectures. In the process, we'll hopefully uncover some new and interesting physics while also considering losses/dissipation in practical transmission lines.

1. A generalized circuit model of a transmission line is shown in Fig. 1(a).

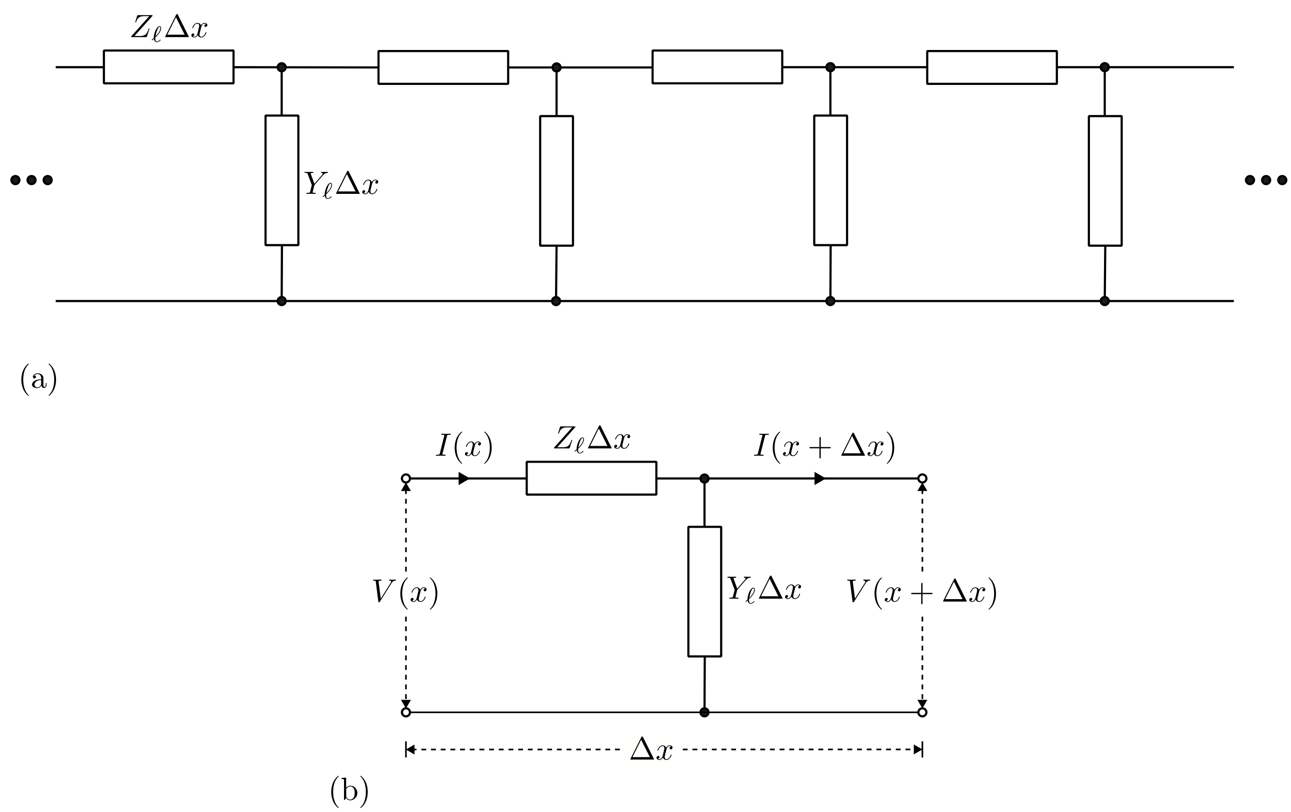

Figure 1: (a) Generalized circuit model of a transmission line. Each elements of the model has a length Δx, series impedance $Z_\ell \Delta x$, and shunt admittance $Y_\ell \Delta x$. (b) A single element of the generalized circuit model shown in (a).

In the figure, Z_ℓ is a per unit length impedance and Y_ℓ is a per unit length *admittance*. The definition of admittance is simply the inverse of an impedance ($Y \equiv 1/Z$). We can recover the model that we analyzed in the lectures by setting $Z_\ell = j\omega L_\ell$ and $Y_\ell = j\omega C_\ell$.

(a) Following the steps used in the September 25 notes, analyze a single cell of the generalized model [Fig. 1(b)] to show that:

$$\frac{d^2V(x)}{dx^2} = Y_\ell Z_\ell V(x) \tag{1}$$
$$\frac{d^2I(x)}{dx^2} = Y_\ell Z_\ell I(x) \tag{2}$$

(b) Next, show that the solution for the voltage amplitude is:

$$V(x) = V_+ e^{-j\beta x} + V_- e^{j\beta x} \tag{3}$$

where:

$$\beta^2 = -Y_\ell Z_\ell \tag{4}$$

2.(a) Let's go ahead and make the substitutions $Z_\ell = j\omega L_\ell$ and $Y_\ell = j\omega C_\ell$ in Eq. (4) to recover the following result that we obtained during lecture:

$$\beta = \pm\omega\sqrt{L_\ell C_\ell} \tag{5}$$

where, in Eq. (5), we have explicitly included both the positive and negative roots of β.

(b) The relationship between β and ω is called the dispersion relation and it can be used to calculate the group velocity v_g and phase velocity v_p of the electromagnetic waves propagating in the transmission line:

$$v_\mathrm{g} = \frac{\partial\omega}{\partial\beta} \tag{6}$$
$$v_\mathrm{p} = \frac{\omega}{\beta} \tag{7}$$

Use Eqs. (6) and (7) to find the group and phase velocities of a conventional transmission line with $Z_\ell = j\omega L_\ell$ and $Y_\ell = j\omega C_\ell$. Based on these results, should we select the positive or negative root of β in Eq. (5)? Explain your reasoning.

(c) The phase velocity velocity is related to the index of refraction n via: $v_\mathrm{p} = c/n$, where c is the vacuum speed of light. For a pair of parallel conductors of radius a separated by distance d, the capacitance and inductance per unit length are given by:

$$C_\ell = \frac{\pi\varepsilon_\mathrm{r}\varepsilon_0}{\ln(d/a)} \tag{8}$$

$$L_\ell = \frac{\mu_\mathrm{r}\mu_0}{\pi}\ln\left(\frac{d}{a}\right), \tag{9}$$

where ε_r and μ_r are the relative permittivity and permeability of the surrounding medium. Use these results to find an expression for the index of fraction. Simplify your answer as much as possible.

3. So far, we've only consider ideal (lossless) transmission lines. We can model the effects of conductor losses by including a per-unit-length resistance R_ℓ in series with L_ℓ [1]. Figure 2 shows a single cell from a transmission line circuit model that includes conductor losses.

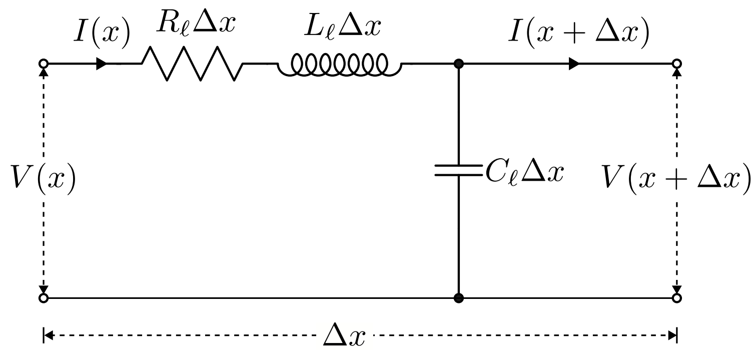

Figure 2: Single cell of the transmission line circuit model that includes conductor losses.

The advantage of the general analysis that we did in question 1 is that we don't need to repeat it – the results given by Eqs. (1)-(4) are still valid. We simply need to update our expressions for Y_ℓ and Z_ℓ.

(a) Write your expressions for Z_ℓ and Y_ℓ. They should be in terms of some or all of the variables: ω, R_ℓ, L_ℓ, and C_ℓ.

(b) Show that, for the lossy transmission line:

$$\beta \approx \omega\sqrt{L_\ell C_\ell} - j\frac{R_\ell}{2}\sqrt{\frac{C_\ell}{L_\ell}} \tag{10}$$

Hint: To derive the approximate expression for β given above, you will need to assume that $R_\ell \ll \omega L_\ell$. In other words, make the realistic assumption that the conductor losses that we are modeling are small.

Notice that including conductor losses in our model has resulted in β becoming a complex quantity. If our solution for β is substituted back into Eq. (3), it results in factors like e^{-x/x_0}, where:

$$x_0 \equiv \frac{2}{R_\ell}\sqrt{\frac{L_\ell}{C_\ell}}.$$

Thus, as expected, lossy transmission lines attenuate signals as they travel the length of line. If you're interested in additional details, including how to include dielectric losses, see Ref. [2].

4. Finally, let's consider a lossless transmission line (i.e. no R_ℓ) in which the positions of the inductors and capacitors have be swapped. Such transmission lines do not occur naturally, but they can be constructed using discrete circuit components (inductors and capacitors). They are called *artificial left-handed* transmission lines. As we'll soon see, they have some interesting/exotic physical properties. Figure 3 shows a single element of such a left-handed transmission line. Notice

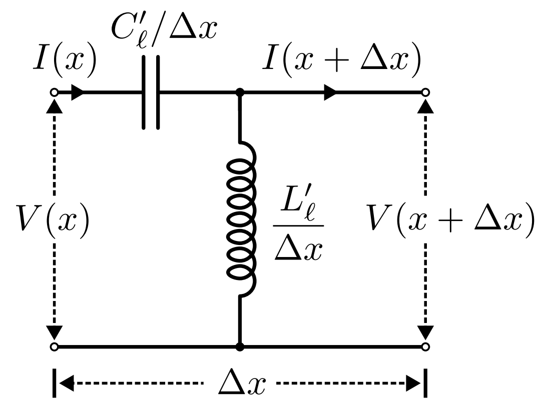

Figure 3: Single cell of an artificial left-handed transmission line.

that for the left-handed transmission line, the capacitance $C = C'_\ell/\Delta x$ and the inductance is given by $L = L'/\Delta x$. In other words, C'_ℓ has units of $\mathrm{F \cdot m}$ and L'_ℓ has units of $\mathrm{H \cdot m}$. These variables have been labeled with primes to remind us that, in this case, they are not per-unit length quantities.

(a) Write your expressions for Z_ℓ and Y_ℓ. They should be in terms of some or all of the variables: ω, L'_ℓ, and C'_ℓ.

(b) Show that, for the left-handed transmission line:

$$\beta = \pm \frac{1}{\omega\sqrt{C'_\ell L'_\ell}} \tag{11}$$

(c) Next, show that the group and phase velocities of the left-handed transmission line are given by:

$$v_{\mathrm{g}} = \mp \frac{1}{\beta^2 \sqrt{L'_\ell C'_\ell}} \tag{12}$$

$$v_{\mathrm{p}} = -v_{\mathrm{g}}. \tag{13}$$

Notice that the group and phase velocities have opposite signs! The group velocity gives the direction of energy flow, so we should select the negative sign for β such that $v_{\mathrm{g}} > 0$ (implying that $v_{\mathrm{p}} < 0$).

(d) Finally, use the phase velocity to find an expression for n. Explicitly check that the units of your expression for n make sense.

Naturally occurring materials have $n > 0$. For this reason, artificial materials engineered to have $n < 0$ (so-called negative index materials, or metamaterials) have drawn a lot of interest from researchers in recent years. If you're interested in learning more about transmission line-based metamaterials, see Refs. [3] and [4].

References

[1] Note that, it is also possible to model the effects of dielectric losses by including a shunt conductance G_ℓ in parallel with C_ℓ.

[2] J. S. Bobowski, "Modeling and measuring the non-ideal characteristics of transmission lines," *Am. J. Phys.*, **89**, 96 (2021).

[3] A. Lai, C. Caloz and T. Itoh, "Composite right/left-handed transmission line metamaterials," *IEEE Microwave Magazine*, **5**, 34 (2004).

[4] G. V. Eleftheriades, "EM transmission-line metamaterials," *Materials Today*, **12**, 30 (2009).